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The data association problem in multi-target tracking has been formulated and solved as a multidimensional assignment problem. Extensive simulations have been performed to demonstrate speed and robustness of these algorithms.

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IN

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by

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Abstract

This report describes some of the problems, achievements, and directions of investigation of three areas of research. The first is the development of combinatorial optimization techniques to solve the central problem of multisensor and multitarget tracking, i.e., the data association problem of partitioning observations into tracks and false alarms. The problem formulation, algorithm design, and real-time solution techniques involve from probability and information theory, system identification, filtering, control systems, combinatorial optimization, and advanced computer architectures, including massively parallel computers. The data association problem for general multitarget/multisensor tracking problems is posed as a class of multidimensional assignment problems. The algorithms under development are based on a recursive Lagrangian relaxation scheme, construct near-optimal solutions in real-time, and use a variety of techniques ranging from two dimensional assignment algorithms, a conjugate subgradient method for the nonsmooth optimization, graph theoretic properties for problem decomposition, and a branch and bound technique for small solution components. A model problem is presented to demonstrate the efficiency and robustness of the current algorithms. The second part centers on investigation of various numerical methods for the solution of nonlinear optimal control problems. The analysis of convergence in infinite dimensional spaces, discretizations, and numerical implementations are in progress for Newton, penalty, augmented Lagrangian, and interior/exterior point methods. The final part is the investigation of parametric constrained optimization problems using numerical bifurcation and continuation methods with applications to design optimization and control systems.

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I. Introduction

This report describes some of the problems, achievements, and directions of investigation of three areas of research as described below. Over the past two years twenty three presentations have been given and eighteen papers have been published or submitted for publication, 2 MS and 4 PhD students have graduated, and A. B. Poore has been appointed associate editor of Computational Optimization and Applications. To reorient this program to the area of multisensor data fusion, Professor Poore spent the summer at Rome Labs as part of the AFOSR Summer Faculty Research Program as described in Section II. The technical information for the last two years is given in Section VII. The three areas of research are briefly explained in the remainder of this introduction.

The first part of this research program centers on the development of combinatorial optimization techniques to solve the central problem of mulitsensor data fusion and multitarget tracking, i.e., the data association problem of partitioning observations into tracks and false alarms. The problem formulation, algorithm design, and real-time solution involve techniques from probability and information theory, system identification, filtering, control systems, combinatorial optimization, and advanced computer architectures, including massively parallel computers. The data association problem for general multitarget tracking problems is formulated as a class of multidimensional assignment problems. The algorithms under development are based on a recursive Lagrangian relaxation scheme, construct high quality suboptimal solutions in real-time, and use a variety of techniques ranging from two dimensional assignment algorithms, a conjugate subgradient method for the nonsmooth optimization, graph theoretic properties for problem decomposition, and a branch and bound technique for small solution components. This problem of partitioning multiple data sets at some cost or to some benefit is also the central problem in perceptual grouping in psychology and stereo correspondence in both biological and computer vision [9]. Thus the applications potentially extend far beyond the current applications. The current status and results of this research effort are described in Section III.

The second part centers on the investigation of various numerical methods for the solution of nonlinear optimal control problems. The analysis of convergence in infinite dimensional spaces, discretizations, and numerical implementations are in progress for Newton, penalty, augmented Lagrangian, and interior point methods. A longer term goal is the investigation of parametric problems in *nonlinear control systems* including but not limited to the nonlinear optimal control problem. Some of the initial results in this direction are described in Section IV.

The final part of this research program is the investigation of parametric nonlinear programming problems using numerical bifurcation and continuation methods with applications to design optimization and parametric control systems, and represents a potential for a real extension of our understanding of basic phenomena, global sensitivity, robustness, and multiplicity of solutions in much the same way that these theoretical and numerical techniques have helped the understanding of dynamical systems and nonlinear equations. Thus the objective in this aspect of the research program is to develop the analytical and numerical techniques to map out regions of qualitatively different behavior and to locate the "stability" boundaries of these regions in parameter space. The latter is important because drastic changes in the optimum occur in the presence of singularities which, in turn, define these "stability" boundaries. Such knowledge allows for the uncertainty in system and model parameters and yields information about the expected behavior when control parameters are varied to enhance the performance of the system under consideration. In addition to providing a global-like sensitivity analysis, these methods are quite efficient in computing multiple optima.

Several model problems taken from the very active area of design optimization are being investigated to test and illustrate the value and applicability of these continuation and bifurcation methods, as well as to provide motivation and focus for further development. Preliminary theory and numerical implementation have been completed as described in detail in Section V.

II. AFOSR Summer Faculty Research Program at Rome Labs, Griffiss, AFB

During the past five years we have worked on multitarget tracking and less on multisensor data fusion. Thus to reorient our research program more in line with the needs of the Department of the Air Force in multisenor data fusion, I spent the summer at Rome Labs as part of the AFOSR summer faculty research program. My goals were: (1) to understand the multisensor data fusion and multitarget tracking problem from the viewpoints of the researchers at Rome Labs, (2) to survey the existing methods and approaches to multisensor data fusion and multitarget tracking, (3) to work on modeling multisensor data fusion and multitarget tracking problems, and (4) to develop a working relation with researchers at Rome Labs with a long term goal of cooperative research. Working with Martin Liggins and Vincent Vannicola at Rome Labs, we: (1) achieved a new and broadened view of the needs in multisensor and multitarget tracking, (2) developed a unifying approach to multisensor data fusion and multitarget tracking based on multidimensional assignment problems, (3) worked on a paper that is to appear [38] and will be presented at the 1993 SPIE meeting in Orlando in April, 1993, (4) arrived at a more unified understanding of the elements needed to smoothly transition 6.1 research into 6.2 and 6.3A areas, and (5) examined the aspects of involving Rome Laboratory personnel in developing an in-house 6.1 research base.

III. Optimization Problems in Multitarget/Multisensor Tracking

This section describes some of the optimization problems in multisensor/multitarget tracking. Section A presents the problem area overview, Section B summarizes the achievements, Section C presents a case study, and Section D gives an overview of the algorithms.

III. A. Problem Statement. The central problem in any multitarget/multisensor surveillance system is the data association problem of partitioning the observations into tracks and false alarms. Current methods [7] for mutitarget tracking generally fall into two categories: sequential and deferred logic. Methods for the former include nearest neighbor, one-to-one or few-to-one assignments, and all-to-one assignments as in the joint probabilistic data association (JPDA) [3]. For track maintenance, the nearest neighbor method is valid in the absence of clutter when there is no track contention, i.e., when there is no chance of misassociation. Problems involving one-to-one or few-to-one assignments are generally formulated as (two dimensional) assignment or multi-assignment problems for which there are some excellent optimal algorithms. This methodology is real-time but can result in a large number of partial and incorrect assignments, particularly in dense or high contention scenarios, and thus incorrect track identification. The difficulty is that decisions, once made, are irrevocable, so that there is no mechanism to correct misassociations. The use of all observations in a scan (e.g., JPDA) to update a track moderates the misassociation problem and has been successful for tracking a few targets in dense clutter.

Deferred logic techniques consider several data sets or scans of data all at once in making data association decisions. At one extreme is batch processing in which all observations (from all time) are processed together, but this is computationally too intensive for real-time applications. The other extreme is sequential processing. Deferred logic methods between these two extremes are of primary interest in this work. The principal deferred logic method used to track large numbers of targets in low to moderate clutter is called multiple hypothesis tracking (MHT) in which one builds a tree of possibilities, assigns a likelihood

score based on Bayesian estimation, develops an intricate pruning logic, and then solves the data association problem by explicit enumeration schemes.

Another important aspect in surveillance systems is the growing use of multisenor data fusion in which one associates reports from multiple sensors together. Once matched, this more varied information has the potential to greatly enhance target identification and state estimation [7]. The central problem is that of data association and the principal method employed is the deferred logic technique of multiple hypothesis tracking [1,2,7,8]. This problem of partitioning multiple data sets at some cost or to some benefit is also the central problem in perceptual grouping in psychology and stereo correspondence in both biological and computer vision [9].

Thus data association and the more general the problem of partitioning multiple data sets to some benefit is a fundamentally important combinatorial optimization problem. These problems generally have the following characteristics: the problems are large scale; the objective function is noisy due to plant noise, errors in the sensor measurements, and modeling uncertainty; and they NP-hard [14] but must be "solved" real-time. Consider the methods currently used to solve these data association problems: explicit enumeration and greedy algorithms. The former is inevitably faulty in dense scenarios since the time required to solve the problem optimally can grow exponentially in the size of the problem. The latter cannot produce near-optimal solutions with any robustness.

Hence the challenge is the design of algorithms that solve these problems to the noise level of the problem in real-time. This is precisely the achievement of this research program. All of these problems have been formulated as multidimensional assignment problems, and a new class of algorithms based on Lagrangian relaxation has been developed to construct near-optimal solutions in real-time. The potential implications are the orders of magnitude improvement in the speed of existing MHT algorithms and the extension of the problems solvable by MHT.

Finally, it is important to note that the use of combinatorial optimization in multitarget tracking is not new and dates back to the mid-sixties and the pioneering work of Sittler[46], who used maximum likelihood estimation to evaluate all possible track updates and employed track splitting (several hypotheses were maintained for each track) and pruning (when their probabilities fell below certain threshold). Maximum likelihood estimation was further investigated by Stein and Blackman [6], who developed a comprehensive probability for track initiation, track length expectancy, missed detections and false alarms. Morefield [25] pioneered the use of the integer programming to solve a set packing problem arising from a data association problem. Multiple hypothesis tracking has been popularized by the fundamental work of Reid [45]. The work here now makes these approaches practical.

III.B. Achievements. Our activities and achievements in 1991 and 1992 have been many and varied and can be briefly summarized as follows:

- As a participant in the AFOSR Summer Faculty Research Program, Professor Poore spent the summer of 1992 at Rome Labs at Griffiss AFB. A major accomplishment was the demonstration the existing deferred logic association techniques such as multiple hypothesis tracking that is the technique used in multisensor data fusion and multitarget tracking can be replaced by multidimensional assignment problems. This work is the subject of two forthcoming papers by Drs. A. B. Poore and N. Rijavec of CSU and Dr. V. C. Vannicola and M. Liggins [37,38] of Rome Labs.
- One class of algorithms for the construction of real-time solutions of the mulitsensor/multitarget tracking problems has been developed. The basic scheme currently [29,40,41] employs preprocessing in the form

of gating and clustering. Then the sparse decomposed problems are solved by a recursive Lagrangian relaxation scheme. A K-dimensional assignment problem is relaxed to a $(K-1)$-dimensional one by incorporating one set of constraints into the objective function using a Lagrangian relaxation of this set. Given a solution of the $(K-1)$-dimensional problem, a feasible solution of the K-dimensional problem is then reconstructed. The $(K-1)$-dimensional problem is solved in a similar manner and the process is repeated until one reaches the two-dimensional problem which is solved exactly. The duality gap in this process is generally small and one obtains good bounds on the optimal solution. The full technical description can be found in the forth coming paper of Poore and Rijavec [29,40]. Other classes of algorithms are underdevelopment.

- Algorithms for an initial tracking system have been developed. This includes problem formulation for track initiation and track maintenance. Algorithms for system optimization and estimation (least squares, Kalman filtering, nonlinear estimation techniques), model simulation, solution quality measurements have been developed and implemented. This provides the basis for checking the quality for different algorithms under development.
- Extensive simulations have been performed to demonstrate the robustness and speed of the assignment solvers. These simulations are the subject of previous and forthcoming publications [29-34,39,41].
- Twenty three presentations have been given and eighteen papers have been submitted or published.

III.C. A Case Study

In this section, a model problem is formulated and solved to demonstrate the overall performance of the algorithms involved.

III.C.1. The Model Problem. One generally assumes that each target is, except in a maneuver, modeled by a state state-space system of the form

$$\begin{aligned} x(k+1) &= F_k(x(k)) + G_k(x(k))w(k) \\ z(k) &= H_k(x(k)) + v(k) \end{aligned}$$

where $x(k)$ is a vector of n state variable, w and v are independent white noise sequences of normal random variables, $z(k)$ represents the measurement at time k associated with this particular target and $H_k(x(k))$ relates the state $x(k)$ to the measurement $x(k)$. In this case study the targets are assumed to travel in two dimensional space according to the constant acceleration model

$$\begin{aligned} x(t,\alpha) &= x_0 + tv_x + \frac{t^2}{2}a_x \\ y(t,\alpha) &= y_0 + tv_y + \frac{t^2}{2}a_y \end{aligned} \tag{C.1}$$

where the parameters in $\alpha = (x_0, v_x, a_x, y_0, v_y, a_y)$ identify a particular target whose track is defined by $p(t,\alpha) \equiv (x(t,\alpha), y(t,\alpha))$.

At a discrete set of *scan times* $\{t_k\}_{k=1}^N$ ($t_1 \le t_2 \le \ldots \le t_N$), a radar located at the origin in this Cartesian space observes error contaminated ranges and angles of the targets in the observation space which is a circle with radius R centered at the origin. Some observations are spurious and some observations of true targets are missed. At time t_k, the radar is assumed to return the set of observations $\{z_{i_k}^k\}_{k=1}^{M_k}$, where M_k is the number of observations and $z_{i_k}^k = (r_{i_k}^k, \theta_{i_k}^k)$. To every scan, a dummy observation z_0^k is added to represent missed detections. Each observation $(r_{i_k}^k, \theta_{i_k}^k)$ arising from an existing target is related to the true observable $H(p(t_k,\alpha))$ by

$$\begin{pmatrix} r_{i_k}^k \\ \theta_{i_k}^k \end{pmatrix} = H(p(t_k,\alpha)) + \begin{pmatrix} e_r^k \\ e_\theta^k \end{pmatrix} \tag{C.2}$$

where e_r^k and e_θ^k are independent zero-mean Gaussian random variables with standard deviations $\sigma_r^k = \sigma_r(t_k)$ and $\sigma_\theta^k = \sigma_\theta(t_k)$, respectively. The measurement error covariance matrix is given by $\Sigma_{r\theta}(t) = diag(\sigma_r^2(t), \sigma_\theta^2(t))$. (More generality is obtained by allowing $\sigma_r(t)$ and $\sigma_\theta(t)$ to vary with the spatial position of the measurement.) The true observable $H(p(t,\alpha))$ is related to the track $p(t,\alpha)$ by

$$H(p(t,\alpha)) = \begin{pmatrix} \sqrt{x(t,\alpha)^2 + y(t,\alpha)^2} \\ \arctan\left[\frac{y(t,\alpha)}{x(t,\alpha)}\right] \end{pmatrix}. \tag{C.3}$$

(If the definition of arctan is based on the principal angle, then the appropriate shifts in $\theta_{i_k}^k$ must be made.) If the observation corresponds to a false alarm or new target, then

$$\begin{pmatrix} r_{i_k}^k \\ \theta_{i_k}^k \end{pmatrix} = \begin{cases} \begin{pmatrix} w_r^k \\ w_\theta^k \end{pmatrix} & \text{if the observation is spurious,} \\ \begin{pmatrix} u_r^k \\ u_\theta^k \end{pmatrix} & \text{if the observation arises from a new source,} \end{cases} \tag{C.4}$$

where the random sequences w^k and u^k have some assumed densities p_f^k and p_ν^k, respectively. A common assumption is that (w_r^k, w_θ^k) and (u_r^k, u_θ^k) are uniformly distributed over the observation space so that

$$p_f^k(w^k) = \begin{cases} \frac{w_r^k}{\pi R^2} & \text{if } 0 \le w_r^k \le R \text{ and } -\pi < w_\theta^k \le \pi \\ 0 & \text{otherwise} \end{cases} \tag{C.5}$$

A similar expression would hold for p_ν^k. The number of false alarms and new targets are assumed to be generated at each time interval $[t_{k-1}, t_k]$ according to a Poisson distributions with expected numbers λ_f^k and λ_ν^k, respectively.

A set of observations $Z_{i_1 \cdots i_N} = \{z_{i_1}^1, \ldots, z_{i_N}^N\}$, containing one observation from each scan, will be called a *track of observations.* Note that some observations in a track of observations might be dummy, representing a missed detection. Tracks of observations $Z_{0 \cdots 0 i_k 0 \cdots 0}$ with a single nonzero index (i.e., a single non-dummy observation) will be taken to represent false reports (clutter).

To determine the most probable partition of the observations into tracks and false reports, each track of observations $Z_{i_1 \cdots i_N}$ must have a likelihood $L_{i_1 \cdots i_N}$ associated with it. First, define $L_{0 \cdots 0} \equiv 1$ and $L_{0 \cdots 0 i_k 0 \cdots 0} \equiv 1$. Next, if $Z_{i_1 \cdots i_N}$ has at least two nonzero indices, the following likelihood expression can be derived [37]:

$$L_{i_1 i_2 \cdots i_N} = \prod_{k=1}^{N} \left\{ P_\phi^k \right\}^{\Delta_{0 i_k}} \left\{ \left[\frac{(1-P_\chi^k) P_d^k p_t^k(z_{i_k}^k | Z_{i_1 \cdots i_N})}{\lambda_f^k p_f^k(z_{i_k}^k | Z_{i_1 \cdots i_N})} \right]^{\delta_{i_k}^k} \left[\frac{\lambda_\nu^k p_\nu^k(z_{i_k}^k | Z_{i_1 \cdots i_N})}{\lambda_f^k p_f^k(z_{i_k}^k | Z_{i_1 \cdots i_N})} \right]^{\nu_{i_k}^k} \right\}^{(1-\Delta_{0 i_k})} \tag{C.6}$$

where

$$P_\phi^k = \begin{cases} P_\chi^k, & \text{if track } Z_{i_1 \cdots i_N} \text{ terminates at scan } k; \\ (1-P_\chi^k)(1-P_d^k), & \text{if track } Z_{i_1 \cdots i_N} \text{ has a missed detection on scan } k; \\ 1, & \text{otherwise,} \end{cases} \tag{C.7a}$$

and the indicator functions ν_i^k, δ_i^k, and Δ_{ij} are defined by

$$\begin{aligned} \nu_i^k \quad &= \begin{cases} 1, & \text{if } z_i^k \text{ is a new target;} \\ 0, & \text{otherwise;} \end{cases} \\ \delta_i^k \quad &= \begin{cases} 1, & \text{if } z_i^k \text{ belongs to an existing track;} \\ 0, & \text{otherwise;} \end{cases} \\ \Delta_{ij} \quad &= \begin{cases} 1, & \text{i = j;} \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{C.7b}$$

Also, P_χ^k and P_d^k denote the probabilities of termination and and detection on scan k. The likelihood functions $p_t^k(z_{i_k}^k|Z_{i_1\cdots i_N})$, $p_f^k(z_{i_k}^k|Z_{i_1\cdots i_N})$, and $p_\nu^k(z_{i_k}^k|Z_{i_1\cdots i_N})$ are those of the error in the observations of the target, false report model, and new source model, respectively.

If the target dynamics (C.1) are known, the likelihod expression in (C.6) can be computed using the densities

$$p_t^k(z_{i_k}^k|Z_{i_1\cdots i_N})) = \frac{1}{2\pi\sigma_r(t_k)\sigma_\theta(t_k)} \exp\left\{ -\frac{1}{2}\left[z_{i_k}^k - H_k(p(t_k,\alpha))\right]^T \Sigma_{r\theta}^{-1}(t_k)\left[z_{i_k}^k - H_k(p(t_k,\alpha))\right]\right\} \qquad (C.8)$$

In the context of formulating the data association problem, however, the track parameters are unknown and must be estimated from the observations $\{z_{i_1}^1,\ldots,z_{i_k}^N\}$. The parameters α are replaced by the maximum likelihood estimate

$$\hat{\alpha} = \text{Arg Max } L_{i_1\cdots i_n}(p(\cdot,\alpha)) = \text{Arg Min } c_{i_1\cdots i_n}(p(\cdot,\alpha)), \qquad (C.9)$$

which can be equivalently characterized as the solution to the nonlinear least squares problem

$$\begin{aligned}\hat{\alpha} &= Arg\ Min\ c_{i_1\cdots i_N}(p(\cdot,\alpha)) \\ &= Arg\ Min \sum_{k=1}^{N}(1-\Delta_{0i_k})\left[z_{i_k}^k - H_k(p(t_k,\alpha))\right]^T \Sigma_{r\theta}^{-1}(t_k)\left[z_{i_k}^k - H_k(p(t_k,\alpha))\right].\end{aligned} \qquad (C.10)$$

III.C.2. The Data Association Problems. This section will address the formulation of the data association problem for two cases. In *track initiation*, no tracks are assumed known. In *track maintenance*, some tracks may be known from prior information. These tracks must be extended using the newly arriving information, while keeping in mind that new tracks might also be initiating. We first address track initiation.

For every track of observations $Z_{i_1\cdots i_N}$, define a 0-1 variable $z_{i_1\cdots i_N}$ via

$$z_{i_1\cdots i_N} = \begin{cases} 1 & \text{observations } \{z_{i_1}^1,\ldots,z_{i_N}^N\} \text{ were generated by the same target} \\ 0 & \text{otherwise} \end{cases} \qquad (C.11a)$$

and a score by

$$c_{i_1\cdots i_N} = -\ln L_{i_1\cdots i_N}, \qquad (C.11b)$$

where $L_{i_1\cdots i_N}$ was defined in (C.6). The requirement that a single non-dummy observation $z_{i_k}^k$ $(1 \le k \le N,$ $1 \le i_k \le M_k)$ be either a false report or assigned to exactly one track can now be expressed as

$$\sum_{i_1=0}^{M_1}\cdots\sum_{i_{k-1}=0}^{M_{k-1}}\sum_{i_{k+1}=0}^{M_{k+1}}\cdots\sum_{i_N=0}^{M_N} z_{i_1\cdots i_{k-1}i_k i_{k+1}i_N} = 1. \qquad (C.12)$$

The data association problem of partitioning the observations into tracks and false reports can now be posed as the following multidimensional assignment problem

$$\begin{aligned} &\text{Minimize} \quad \sum_{i_1=0}^{M_1}\cdots\sum_{i_N=0}^{M_N} c_{i_1\cdots i_N} z_{i_1\cdots i_N} \\ &\text{Subject To}: \quad \sum_{i_2=0}^{M_2}\cdots\sum_{i_N=0}^{M_N} z_{i_1\cdots i_N} = 1, \quad i_1 = 1,\ldots,M_1, \\ &\sum_{i_1=0}^{M_1}\cdots\sum_{i_{k-1}=0}^{M_{k-1}}\sum_{i_{k+1}=0}^{M_{k+1}}\cdots\sum_{i_N=0}^{M_N} z_{i_1\cdots i_N} = 1, \\ &\quad\text{for} \quad i_k = 1,\ldots,M_k \text{ and } k = 2,\ldots,N-1, \\ &\sum_{i_1=0}^{M_1}\cdots\sum_{i_{N-1}=0}^{M_{N-1}} z_{i_1\cdots i_N} = 1, \quad i_N = 1,\ldots,M_N \\ &z_{i_1\cdots i_N} \in \{0,1\} \text{ for all } i_1,\ldots,i_N, \end{aligned} \qquad (C.13)$$

Next, *track maintenance* using a sliding window is developed. Suppose that the observations on P *previous* scans (of observations) have been partitioned into tracks and false alarms and that K *new* scans of observations are to be added. One approach to solving the resulting data association problem is formulate the problem as a track initiation problem with $P+K$ scans. This is the previously mentioned *batch* approach. The *deferred logic approach* adopted here is to treat the track extension problem within the framework of a window sliding over the observation sets. First assume that the scans of observations are partitioned into three components: D *discarded* scans of observations, R *retained* scans of observations from the P *previously* processed scans, and K *new* scans of observations. Thus the number of scans in the sliding window is $N = R + K$ while the number of *discarded* scans is $D = P - R$.

Let M_0 denote the number of confirmed tracks previously constructed from the discarded and retained regions that are present at the start of the tracking window. (These M_0 tracks may be obtained as the solution of a previous problem assignment problem, may be union of the best K such solutions, or may be all of the feasible tracks. However, tracks terminated in the discarded region are generally not included in M_0.) Suppose the i_0^{th} such track is denoted by T_{i_0} for $i_0 = 1, \ldots, M_0$. For $i_0 > 0$, the $(N+1)$-tuple $\{T_{i_0}, z_{i_1}^1, \ldots, z_{i_N}^N\}$ will denote a track T_{i_0} plus a set of observations or measurements $\{z_{i_1}^1, \ldots, z_{i_N}^N\}$, actual or dummy, that are feasible with the track T_{i_0}. The $(N+1)$-tuple $\{T_0, z_{i_1}^1, \ldots, z_{i_N}^N\}$ will denote a track that initiates in the sliding window.

Analogous to the track initiation case, one can define the zero-one variable

$$z_{i_0 i_1 \cdots i_N} = \begin{cases} 1 & \text{if } \{T_{i_0}, z_{i_1}^1, \ldots, z_{i_N}^N\} \text{ is assigned as a unit,} \\ 0 & \text{otherwise.} \end{cases} \tag{C.14a}$$

and the corresponding cost for the assignment of the sequence $\{T_{i_0}, z_{i_1}, \ldots, z_{i_N}\}$ to a track by

$$c_{i_0 i_1 \cdots i_N} = -\ln L_{i_0} L_{i_1 \cdots i_N}. \tag{C.14b}$$

Here $L_{T_{i_0}}$ is the composite likelihood from the discarded scans just prior to the first scan in the window for $i_0 > 0$, $L_{T_0} = 1$, and $L_{i_1 \cdots i_N}$ is defined as in (C.6) for the N-scan window. ($L_{T_0} = 1$ is used for any tracks that initiate in the sliding window.) The data association problem for track maintenance can thus be formulated as

$$\begin{aligned}
\text{Minimize} \quad & \sum_{i_0=0}^{M_0} \cdots \sum_{i_N=0}^{M_N} c_{i_0 \cdots i_N} z_{i_0 \cdots i_N} \\
\text{Subj. To} \quad & \sum_{i_1=0}^{M_1} \cdots \sum_{i_N=0}^{M_N} z_{i_0 \cdots i_N} = 1, \quad i_0 = 1, \ldots, M_0, \\
& \sum_{i_0=0}^{M_0} \sum_{i_2=0}^{M_2} \cdots \sum_{i_N=0}^{M_N} z_{i_0 i_1 \cdots i_N} = 1, \quad i_1 = 1, \ldots, M_1, \\
& \sum_{i_0=0}^{M_0} \cdots \sum_{i_{k-1}=0}^{M_{k-1}} \sum_{i_{k+1}=0}^{M_{k+1}} \cdots \sum_{i_N=0}^{M_N} z_{i_0 \cdots i_N} = 1, \\
& \qquad \text{for} \quad i_k = 1, \ldots, M_N \text{ and } k = 2, \ldots, N-1, \\
& \sum_{i_0=0}^{M_0} \cdots \sum_{i_{N-1}=0}^{M_{N-1}} z_{i_0 \cdots i_N} = 1, \quad i_N = 1, \ldots, M_N, \\
& z_{i_0 \cdots i_N} \in \{0, 1\} \text{ for all } i_0, \ldots, i_N.
\end{aligned} \tag{C.15}$$

Note that the association problem involving N scans of observations is an N-dimensional assignment problem for track initiation and an $(N+1)$-dimensional one for track maintenance.

III.C.3. Numerical Simulations. The performance of the tracking algorithms presented in this work depends on many factors, including target density (i.e., the number of targets per unit space), the space size, the measurement error covariances and probabilities of the detection and false report rates. This section presents results of two studies investigating the impact of changing the sensor error characteristics and window sizes on the performance of these algorithms. Results of more comprehensive parametric studies will be presented in a future work.

The tracking problems considered in both studies had the following characteristics: the observation space is circular with a radius of 20 miles and sensor in the center and 10 targets that were initiated before the first scan and never terminated (targets were generated so as to never leave the space). The initial target speeds were between 200 and 900 miles per hour and the target acceleration was not more than 0.0034 miles per second squared (or 44,064 miles per hour squared). The scan times were every 10 seconds, and fifteen scans of observations were used. Note that the total gain in velocity due to acceleration was thus not more than 1500 mph. The sensor returned measurements in polar coordinates, as described in Section III.C.2. All the observations in each scan were synchronous. The range error was relative, the angle error was absolute, and neither varied with time. Both the missed detections and false reports were allowed by the sensor.

The problems were generated randomly To obtain a meaningful sample for the comparisons, a set of 100 problems for each set of parameters was generated, differing only in the random number generator seed. All results presented in this section are thus averages over 100 problems.

The first study investigated the impact of changing the standard deviations of the measurement errors. All the problems in the first study had the probability of detection of 0.95, and the probability of false reports (i.e., the probability that an arbitrary observation was false report) of 0.05. A six scan window was used for tracking. Thus a six dimensional assignment problem governed the data association problem for track initiation and a seven dimensional one for track maintenance. The data association problems arising from the tracking problems in this study had between 350 and 5000 variables and were on the average solved in less than half a second for the biggest problems, using an IBM RS/6000-550 workstation. Table 1 shows the quality of the computed tracking problem solution for each of nine combinations of the range and angle errors. The solution quality was checked after each scan in the simulations. Each scan thus corresponds to a column in the table. The column for 6 scans refers to track initiation. All other columns are results for track maintenance. The angle error standard deviation σ_θ is expressed in degrees.

σ_r	σ_θ	6 sc.	7 sc.	8 sc.	9 sc.	10 sc.	11 sc.	12 sc.	13 sc.	14 sc.	15 sc.
0.01	0.50	99.0	99.7	99.6	99.9	99.9	99.9	99.9	99.9	100.0	100.0
0.01	1.00	99.4	99.5	99.2	99.7	99.9	99.9	99.9	100.0	99.9	100.0
0.01	1.50	98.8	99.7	99.8	99.9	99.8	99.9	99.7	100.0	100.0	100.0
0.02	0.50	98.9	99.3	99.3	99.5	99.9	99.9	99.9	100.0	100.0	100.0
0.02	1.00	98.9	99.6	99.7	99.9	99.9	99.9	99.8	100.0	100.0	100.0
0.02	1.50	99.1	99.5	99.7	99.4	99.6	99.9	99.8	100.0	100.0	100.0
0.03	0.50	98.1	98.7	99.5	99.2	99.4	99.7	99.7	99.4	100.0	99.9
0.03	1.00	97.8	98.9	98.7	99.7	99.7	99.6	99.8	100.0	100.0	99.8
0.03	1.50	98.2	98.8	99.2	99.5	99.8	99.8	99.8	99.8	99.9	99.9

Table 1 : Solution quality for varying measurement error

Table 1 shows that the quality of the solution increases as more information becomes available and the track estimates become better. Six scan window seems adequate for tracking problems of this level of noise, especially since our studies of individual problems indicate that the quality criteria described in Section 4

are a little too stringent in the sense that tracks proclaimed "missed" often lie just outside of the region defined by the quality criteria.

For the second study, the measurement error standard deviations were kept constant at $\sigma_r = 0.01$ and $\sigma_\theta = 0.5°$, while the probability of detection was either 0.95 or 0.7 and the probability that an arbitrary report was false was either 0.05 or 0.3. For each of the four combinations of these two parameters, tracking problems were solved using 4, 5, 6, and 7 scan moving windows. Assignment problem sizes ranged from 30 to 2500 variables and longest solution times were 0.54 seconds for largest windows and highest noise levels (most of the cases had the solution times of less than 0.3 seconds). Table 2 shows the solution quality for all 16 combinations of probability of detection (PD), probability of false reports (PFR) and window size (Win.). Inapplicable entries in Table 2 are denoted by "-". As in Table 1, the results are presented on scan by scan basis, with the first number in each row indicating track initiation, while the remaining numbers refer to track maintenance.

PD	PFR	Win	4 sc.	5 sc.	6 sc.	7 sc	8 sc.	9 sc.	10 sc.	11 sc.	12 sc.	13 sc.	14 sc.	15 sc.
0.70	0.05	4	25	35	40	43	45	44	48	47	48	47	46	46
0.70	0.05	5	-	50	63	70	73	75	76	76	77	75	75	75
0.70	0.05	6	-	-	72	83	87	89	89	90	90	89	89	88
0.70	0.05	7	-	-	-	87	93	96	97	97	97	96	95	95
0.70	0.30	4	23	33	38	44	46	46	47	46	46	47	[illegible]	49
0.70	0.30	5	-	49	61	68	73	74	76	76	77	75	75	75
0.70	0.30	6	-	-	70	81	86	89	90	90	90	91	90	89
0.70	0.30	7	-	-	-	81	90	92	94	96	96	96	96	95
0.95	0.05	4	81	86	91	94	96	96	97	97	97	97	97	97
0.95	0.05	5	-	98	99	99	99	100	100	100	100	100	100	100
0.95	0.05	6	-	-	99	100	100	100	100	100	100	100	100	100
0.95	0.05	7	-	-	-	100	100	100	100	100	100	100	100	100
0.95	0.30	4	80	87	91	94	97	97	98	98	98	97	97	97
0.95	0.30	5	-	97	98	98	99	100	100	100	100	100	100	100
0.95	0.30	6	-	-	96	99	99	100	100	100	100	100	100	100
0.95	0.30	7	-	-	-	99	99	100	100	100	100	100	100	100

Table 2 Solution quality for different PD, PFR and window sizes

Table 2 shows that the quality of the solution again increases as more information becomes available. However, in problems with the low probability of detection, even after 15 scans of information, the algorithm that uses only four scans in the moving window fails to identify over half the tracks. As the window increases, so does the quality, since more information is available to the tracking algorithm. The capability to vary the window sizes is thus crucial if the algorithm is to handle problems with different noise levels successfully. Even though the model initiates all the targets before the first scan is made, the results in Table 2 show that track initiation must be allowed on later scans, as outlined in Section 3. Especially in problems with low probability of detection, some targets will not be indentified in the first few windows, simply because they have not been detected enough times, and are thus initiated later.

Comparing Tables 1 and 2, it is obvious that varying the probability of detection has more impact on the solution quality than varying the false report rate or the measurement rate. This is not really surprising since lowering the probability of detection actually removes information from the problem, while increasing the false report rate and increasing the measurement errors just adds noise. However, using the appropriate

window sizes, the algorithms presented in this work construct high quality solutions even for very noisy tracking problems.

Finally, it should be pointed out that even the limited parametric studies presented in this section involved solving tracking problems with widely differing amounts and types of noise. The only adjustment made to the algorithms was the size of the sliding window. This indicates that the algorithms for solving tracking problems using multiscan sliding windows, maximum likelihood estimation, and Lagrangian relaxation for data association problems, are quite robust and thus likely to be effective for a wide range of tracking problems.

III. D. Algorithm Overview. A primary objective of this work has been the development of algorithms for the fast construction of high quality (near-optimal) suboptimal solutions of the following multidimensional assignment problem (C.13). These assignment problems, as developed in Section III.C, possess the following important characteristics: the problem is large scale; the objective function is noisy due to plant noise, errors in the sensor measurements, and modeling uncertainty; it is NP-hard[12] but must be solved real-time. Gating and clustering techniques are generally used to reduce the size and complexity of the problem, thereby making the problems sparse. We argue that the problem should be solved to the noise level and not to optimality, since the objective is to use this assignment problem as a vehicle to identify objects in sensor fusion and estimate tracks in tracking. The NP-hardness and real-time needs rule out conventional techniques such as branch and bound or explicit enumeration. The sparsity of the problem raises the issue of whether or not (D.1) has a feasible solution. To resolve this we assume that all zero-one variables with exactly one nonzero index are free to be assigned and the corresponding cost coefficients are defined. The zero-one variable $z_{0 \cdots 0}$ and the corresponding term $c_{0 \cdots 0} z_{0 \cdots 0}$ in the objective function are present for notational convenience. Finally, other problems of interest include the situation in which the " $= 1$" in the constraint (D.1) is changed to " $\leq$, $=$, or $\geq n^k_{i_k}$" for some nonnegative integer $n^k_{i_k}$. However, we shall not address these problems.

The algorithm development in this work is based on Lagrangian relaxation, which originally gained prominence as a method for efficiently obtaining tight bounds for a branch and bound algorithm in Held and Karp's highly successful work on the traveling salesman problem. Overviews of this methodology can be found in the works of Geoffrion, Fisher, Shapiro, the book by Nemhauser and Wolsey, and the references therein. The particular algorithm developed in this work is motivated by that of Frieze and Yadegar for three dimensional assignment problems; however, an overview of the algorithms developed in this work is perhaps more easily described interms of a prototype algorithm for a general integer programming problem.

Consider the integer programming problem

$$
\begin{aligned}
\text{Minimize} \quad & c^T z = V(z) \\
\text{Subject To} \quad & Az \geq b \\
& Bz \geq d \\
& z_i \text{ is an integer for } i \in I.
\end{aligned}
\tag{D.2}
$$

where the partitioning of the constraints is natural in some sense. The Lagrangian relaxation of (D.2) relative to the constraints $Bz \geq d$ is defined to be

$$
\begin{aligned}
\Phi(u) = \text{Minimize} \quad & \phi(z,u) \equiv \{c^T z - u^T (Bz - d)\} \\
\text{Subject To} \quad & Az \geq b \\
& z_i \text{ is an integer for } i \in I \\
& u \geq 0
\end{aligned}
\tag{D.3}
$$

where $u \geq 0$ is interpreted componentwise. If the constraint set $Bz \geq d$ is replaced by $Bz = d$, the nonnegativity constraint on u is removed. $\mathcal{L} = c^T z - u^T(Bz - d)$ is the Lagrangian relative to the constraints $Bz \geq d$, and hence the name *Lagrangian relaxation*. Next, if $\bar{z}$ is an optimal solution to (D.2), problems (D.2) and (D.3) imply

$$\Phi(u) \leq V(\bar{z}) \quad \text{for all } u \geq 0. \tag{D.4}$$

Given a specific multiplier u, let $z_r(u)$ and $\bar{z}_r(u)$ denote suboptimal and optimal solutions of the relaxed problem (D.3), respectively. Generally, $z_r(u)$ is not feasible for the relaxed constraint $Bz \geq d$; however, if $\bar{z}_r(u)$ *is* feasible, then it is then also optimal for (D.2). Thus one must develop a recovery procedure for constructing a feasible solution of (D.2) from either of $z_r(u)$ or $\bar{z}_r(u)$. There are several reasons why the resulting feasible solution z_f might be a good solution of (D.2). First, if the multiplier $u = \bar{u} \geq 0$ is chosen as the maximizer of the problem

$$\text{Maximize } \{\Phi(u) \ : \ u \geq 0\} \tag{D.5}$$

and the duality gap $[\Phi(\bar{u}), V(\bar{z})]$ is small, then the recovered feasible solution z_f of (D.2) from the solution $z_r(\bar{u})$ of (D.3) may be close to $\bar{z}$. (The experience of many researchers is that this duality gap tends to be much smaller for equality constrained problems than for corresponding inequality constrained problems[10].) Secondly, the term $-u^T(Bz - d)$ in (D.3) acts like a penalty for violating the constraint $Bz - d \geq 0$, thereby forcing $z_r(\bar{u})$ closer to the optimal solution $\bar{z}$ of (D.2). Finally, the recovery procedure should be designed to minimize any remaining flexibility in the objective function in (D.1). Thus given this rationale, the following prototype algorithm abstracts some of the ideas of the work on three dimensional assignment problems by Frieze and Yadegar:

Prototype Algorithm. *Construct a sequence of multipliers $\{u_k\}_{k=0}^{\infty}$ in the course of maximizing $\Phi(u)$ defined in (D.3) and a corresponding sequence of feasible solutions $\{z_k\}_{k=0}^{\infty}$ of (D.2) as follows:*

A. *Choose an initial approximation u_0.*

B. *Given u_k, determine a new multiplier u_{k+1} from a step in the maximization problem (D.5), so that $\Phi(u_k) \leq \Phi(u_{k+1})$.*

C. *Given u_{k+1} and a solution $z_r(u_{k+1})$ of (D.3), recover a feasible solution z_{k+1} of the integer programming problem (D.2).*

In the absence of any a prior knowledge of the initial multiplier u_0, a good neutral choice in Part A is $u_0 = 0$. Part B of this algorithm is the nonsmooth optimization phase and one of the most widely used methods for non-smooth optimization is the subgradient algorithm, which is the nonsmooth analog to the steepest ascent method. Analogous to conjugate gradient methods for smooth optimization is the class called "bundle methods". This includes the space dilation method of Shor, the "bundle-trust region" method due to Schramm and Zowe, and the conjugate subgradient method of Wolfe. Wolfe's algorithm is used in this work. The recovery procedure is part C of this algorithm and can vary considerably with the problem. Note that $\Phi(u_k) \leq \Phi(\bar{u}) \leq V(\bar{z}) \leq V(z_k)$, so that we have a bound on the optimal solution. With an estimate of the noise level in the problem, we can then use these bounds as a stopping criteria. The explicit developed for the multidimensional assignment problems are presented in [29] and forthcoming work.

IV. Infinite Dimensional Optimization Problems and Numerical Control

The classes of optimal control problems currently under investigation are subclasses of the problem

$$
\begin{aligned}
\text{Minimize} \quad & J[x,u] := \varphi(t_0, t_1, x(t_0), x(t_1)) + \int_{t_0}^{t_1} f_0(t, x, u)\,dt \\
\text{Subject To} \quad & \dot{x} = f(t, x(t), u(t)) \\
& B(t_0, t_1, x(t_0), x(t_1)) = 0 \\
& h(t, x(t), u(t)) = 0 \text{ a.e. } [t_0, t_1] \\
& g(t, x(t), u(t)) \leq 0 \text{ a.e. } [t_0, t_1] \\
& u(t) \in \Omega \text{ a.e. } [t_0, t_1] \\
& \mathcal{F}_i(t_0, t_1, x(t_0), x(t_1), x, u) \leq 0 \text{ for } i = 1, \ldots, k \\
& (x, u) \in W^{1,\infty}([t_0, t_1], \mathbb{R}^n) \times L^{\infty}([t_0, t_1], \mathbb{R}^m)
\end{aligned}
$$

where x is an n-vector, u is an m-vector, B is a boundary operator, Ω is a closed convex set, $\mathcal{F}_i$ is a functional and $W^{1,p}([t_0, t_1], \mathbb{R}^n)$ is the usual Sobolev space which can be characterized via the Sobolev imbedding theorem as consisting of those absolutely continuous vector functions with the first derivative in $L^p([t_0, t_1], \mathbb{R}^n)$. The functions φ, f_0, f, B, h, $\mathcal{F}_i$, and g are assumed to be at least C^2 with respect to their arguments.

Our interest in this problem is two fold. First, working with former PhD student B. Yang, Professor W. W. Hager of the University of Florida and Professor Asen Dontchev of Bulgiarian Academy of Sciences and Math Reviews, we have investigated the convergence of various numerical methods (Newton's, penalty, augmented Lagrangian, interior point methods) in the appropriate infinite dimensional spaces. This work has evolved as follows: Poore, Yang, and Hager [42] have investigated convergence of penalty, multiplier, and Newton methods for a subclass of the above problems without set constraints and inequality constraints on the controllers. A more theoretical analysis of the above problem, again without set constraints or inequality pointwise constraints on the state variables and controller was developed in the PhD thesis of B. Yang [50]. This latter work has been considerably generalized to include these pointwise equality and inequality constraints in the recent work of Dontchev, Hager, Poore, and Yang [11]. The approach was to first derive sufficient optimality conditions for an infinite dimensional optimization problem in a setting that is applicable to optimal control problems with endpoint constraints and with equality and inequality constraints on the controls. Under the hypotheses of the sufficient optimality theorem we show that the solution to an optimal control problem possesses a Lipschitz stability property relative to problem perturbations. As an application of this stability result, we establish convergence results for the successive quadratic programming algorithm and for penalty and multiplier approximations applied to optimal control problems.

The second area of research interest is the parametric problem associated with the above optimal control problem. The interaction of multiple and bifurcating states in the absence of controls, periodic phenomena, chaotic behavior, and bifurcating controls arising from the dynamical systems and holonomic constraints is open to investigation. Given a certain phenomena arising from a dynamical system, the problem may be to control this phenomena, to determine multiple solutions, or to investigate the dependence of a solution on the system parameters over a wide range, i.e., global sensitivity. (The latter is also important in adaptive control.) The development and use of theoretical and numerical bifurcation and continuation methods in dynamical systems and nonlinear equations has been spectacularly successful in analyzing and understanding the phenomena represented by these systems, but we know of no systematic treatment or works on the

constrained nonlinear parametric control problem paralleling that found in dynamical systems. Thus a long term goal of this research program will be the investigation of parametric problems in *nonlinear control systems* including but not limited to the nonlinear optimal control problem.

V. Parametric Nonlinear Programming and Control

This section describes the parametric optimization problem, some of the accomplishments over the last two years and an application in design optimization.

V.A Problem Statement. The use of bifurcation and singularity theory in the investigation of parametric problems in optimization and control represents a potential for a real extension of our understanding of basic phenomena, global sensitivity, robustness, and multiplicity of solutions in both finite and infinite dimensional optimization problems, in much the same way that these theoretical and numerical techniques have helped our understanding of dynamical systems and nonlinear equations. Thus the objective in this program has been the development of the analytical and numerical techniques to map out regions of qualitatively different behavior and to locate the "stability" boundaries of these regions in parameter space. The latter is important because drastic changes in the optimum occur at singularities, which define these "stability" boundaries. Our initial work [28,35,47,48] has been the classification and analysis of singular points in the nonlinear parametric programming problem. Numerical techniques for predictor-corrector continuation techniques have been developed using a nonstandard variable order Adams-Bashforth predictors with an adaptive error-step size control [19]. This software, which is available through Netlib, is particularly efficient and robust for parametric problems. Numerical continuation and bifurcation techniques are being developed and tailored to the finite dimensional constrained optimization problem in support of future work on large scale control systems design optimization [20,23,24]. In the remainder of this section, the parametric nonlinear programming problem is defined and some of the accomplishments during 1991 and 1992 are presented.

Mathematically, the parametric nonlinear programming problem is that of determining the behavior of solution(s) as a parameter or vector of parameters $\alpha \in \mathbb{R}^r$ varies over a region of interest for the problem

$$\begin{aligned} \text{Minimize } \{f(x,\alpha) \mid c_i(x,\alpha) &= 0 \quad \text{for} \quad i \in E \\ c_i(x,\alpha) &\leq 0 \quad \text{for} \quad i \in I\} \end{aligned} \tag{A.1}$$

where $E = \{1,\dots,p\}$ and $I = \{p+1,\dots,p+q\}$ represent the index sets for the equality and inequality constraints, respectively, and where $f : \mathbb{R}^{n+r} \to \mathbb{R}$, $c_E : \mathbb{R}^{n+r} \to \mathbb{R}^q$ and $c_I : \mathbb{R}^{n+r} \to \mathbb{R}^p$ are assumed to be at least twice continuously differentiable. Using first-order necessary conditions as motivation, one can convert the characterization of a solution to this problem as a solution of a system of nonlinear equations. At a regular point of this latter system, the implicit function theorem rigorously justifies the computation of the derivatives of the primal and dual variables with respect to the parameter α. These derivatives provide the basis for local sensitivity analysis as presented in the work of Fiacco [12,13] and references therein. Thus all the "action" is at the singular points of this system (A.1) where catastrophic failure, extreme sensitivity, and jumps to undesirable operating states can occur. We have investigated these singularities in several papers [28,35,47,48] via bifurcation and singularity theory. (These singular points are characterized by a loss of strict complementarity, a violation of the linear independence constraint qualification, or the singularity of the Hessian of the Lagrangian on the tangent space to the active constraints.)

The work in the last two years has turned to the development of numerical continuation and bifurcation techniques for the systematic exploitation of these methods in applied problems. We now give a synopsis of the algorithms and methods that can be found in a series of papers by Poore and Lundberg [19-24].

A solution of the parametric programming problem (A.1) is a solution of the following system of nonlinear equations

$$F(x,\lambda,\nu;\alpha) = \begin{bmatrix} \nabla_x \mathcal{L}(x,\lambda,\nu;\alpha) \\ \Lambda c(x;\alpha) \\ \nu^2 + \lambda^T\lambda - \beta_0^2 \end{bmatrix} = 0 \tag{A.2}$$

where $\mathcal{L} = \mathcal{L}(x,\lambda,\nu) = \nu f(x) + \sum_{i=1}^{p+q} \lambda_i c_i(x)$ is the Lagrangian and Λ is a diagonal matrix with $\Lambda_{ii} = 1$ for $i \in E$ and $\Lambda_{ii} = \lambda_i$ for $i \in I$. Let $\mathcal{A} = E \cup \{i \in I : c_i(x,\alpha) = 0\}$ denote the active set. Note that any solution of this system at which $\nu \geq 0$, $c_i(x) \leq 0$, and $\lambda_i \geq 0$ for all $i \in I$ satisfies the Fritz John first-order necessary conditions. This system also employs a nonstandard normalization $\nu^2 + \lambda^T\lambda - \beta_0^2 = 0$, where β_0 is a fixed positive real number. The standard normalization $\nu = 1$ is not employed since it requires a constraint qualification for its validity and the violation of the linear independence constraint qualification is a singularity in the above system (A.2).

Since a multiplier corresponding to an inactive constraint is zero, the system (A.2) can be reduced in complexity by using an *active set strategy*. The inactive constraints, i.e., those c_i for which $i \in I - \mathcal{A}$, are thus removed, yielding the active set system

$$\bar{F}(z,\alpha) = \begin{bmatrix} \nabla_x \mathcal{L}(z,\alpha) \\ \bar{c}(x,\alpha) \\ B(\lambda,\nu) \end{bmatrix} = 0, \quad \text{where} \quad z = \begin{bmatrix} x \\ \lambda \\ \nu \end{bmatrix} \in \mathbb{R}^m, \tag{A.3}$$

$m = n + |\mathcal{A}| + 1$, $\lambda = (\lambda_1, \ldots, \lambda_p, \lambda_{i\in\mathcal{A}\cap I})$, and $\bar{c} = (c_1, \ldots, c_p, c_{i\in\mathcal{A}\cap I})$, $\mathcal{L}(z,\alpha) = \nu f(x,\alpha) + \sum_{i\in\mathcal{A}} \lambda_i c_i(x,\alpha)$ and $B(\lambda,\nu) = \nu^2 + \lambda^T\lambda - \beta_0^2$. Continuation for the system (A.3) along with locating the zeros in one or more of the active, inequality multipliers λ_i, $i \in \mathcal{A} \cap I$ or an inactive constraint c_i for $i \in I - \mathcal{A}$ and changing the active set appropriately is then equivalent to continuation for the full system (A.2).

V.B Status of the Algorithms. Numerical algorithms for numerical linear algebra in the continuation procedure, critical point type, singularity detection and classification, and branch switching have been developed in three papers of Poore and Lundberg [20,23,24] with additional work in progress. We now give a brief overview of these results.

The linear systems that arise in the continuation steps can be reduced via block elimination algorithms [6] to the solution of several linear systems of the form

$$W \begin{bmatrix} \Delta x \\ \Delta\lambda \end{bmatrix} = \begin{bmatrix} g \\ b \end{bmatrix} \quad \text{where} \quad W = \begin{bmatrix} H & A \\ A^T & 0 \end{bmatrix}, \tag{B.1}$$

the matrix H is the Hessian of the Lagrangian or some approximation to it, and $A^T = D_x\bar{c}(x,\alpha)$. During the continuation steps the matrix H need not be positive definite on the tangent space to the active constraints; however, both null and range space methods are easily modified to form the basis for the linear algebra steps[20].

The classification of critical point type is based on [20]

$(B.2a)$	sign ν,
$(B.2b)$	signs of $c_i(x,\alpha)$ for $i \in I - \mathcal{A}$ and λ_i for $i \in I \cap \mathcal{A}$,
$(B.2c)$	signs of the eigenvalues of $\nabla_x^2 \mathcal{L}_T$,

where $\nabla_x^2 \mathcal{L}_T$ denotes the restriction of the Hessian of the Lagrangian to the tangent space of the active constraints $\mathcal{N}(A^T)$. It is only the latter class of signs (B.2c) that require computation, and this can be

accomplished by computing the inertia of the reduced Hessian, which can be accomplished by either null or range space methods [20].

Methods for *detecting singularities* due to the loss of strict complementarity, loss of the linear independence constraint qualification, and singularity of the Hessian of the Lagrangian on the tangent space to the active constraints have been extensively developed in the work of Lundberg and Poore [20]. The philosophy behind singularity detection is to skip over them during the continuation procedure, detect their presence, and then take the appropriate action, e.g., switch branches, change orientation, or continue along the current branch. The detection and classification for simple bifurcations and folds have been show to be inexpensive 'by-products' of the continuation procedure [20]; however, due to the technically detailed classification we omit a further discussion of this problem.

V.C. A Model Problem from Design Optimization. The numerical continuation techniques described in the previous sections will now be used to obtain a "global" analysis of the sensitivity, stability, and multiplicity of minima for a parametric nonlinear programming problem arising from design optimization. The problem, which is simple yet still exhibits the basic phenomena, involves the design of a two bar planar truss with semi-span 1, unloaded height h, and load p as indicated in Figure 1.

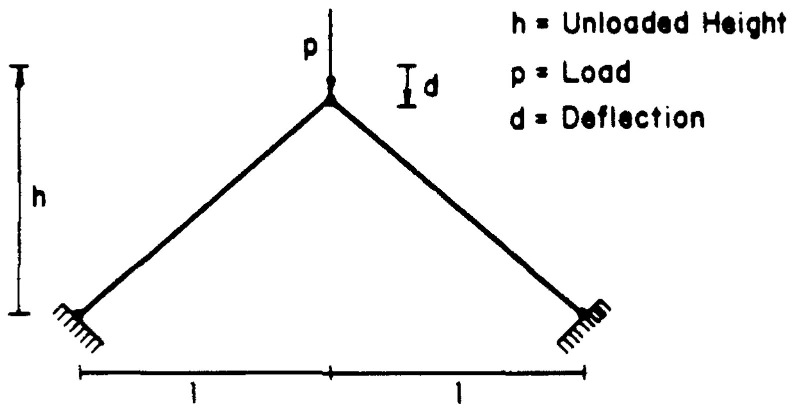

Figure 1: Loaded Two Bar Truss

Given a specific unloaded height h and load p, the deflection d is a minimizer of the potential energy $E(d,h;p) = -pd + \left(\sqrt{1+h^2} - \sqrt{1+(h-d)^2}\right)^2 / \sqrt{1+h^2}$. Rheinboldt [44] used this model problem to illustrate continuation methods in structural analysis and has given a rather complete solution to both the static and parametric problems. Rao and Papalambros [43] posed a corresponding optimal design problem as that of choosing the height h to minimize the deflection subject to $0 \le h \le 1.5$. This problem is posed mathematically as

$$(C.1) \qquad \begin{array}{ll} \text{Minimize} & d \\ \text{Subject To} & \nabla_d E(d,h;p) = 0 \\ & 0 \le h \le 1.5 \end{array}$$

In addition to selecting the minimizer, the state (d,h) must also be selected so that the potential energy $E(d,h;p)$ is minimized with respect to d. The corresponding parametric problem is to determine the solution and its properties as the load p varies over all physically important ranges. The numerical methods discussed in the work of Lundberg and Poore and briefly discussed in the previous subsection were used to obtain a

global solution of this problem as shown in Figure 2.

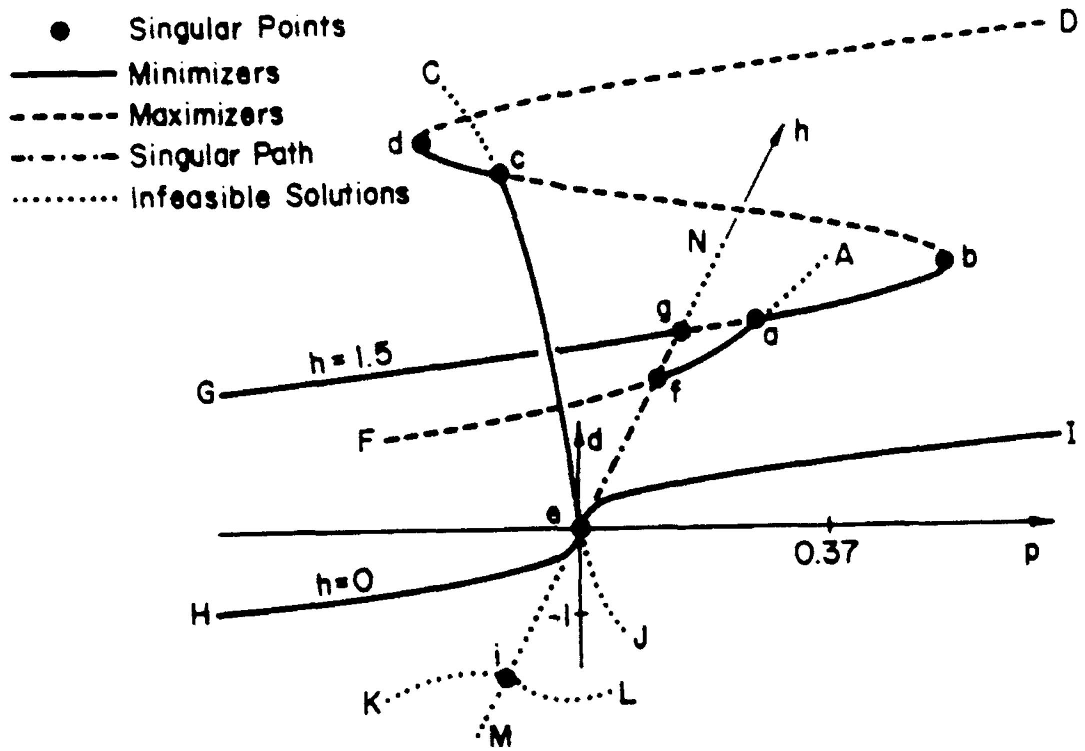

Figure 2: Solutions of (A.2) for Problem (C.1)

Here, the displacement d and unloaded height h as p varies and represents a projection of the solutions of (C.1) into (h, d, p) space. Solid and dashed lines indicate paths of local minimizers and maximizers, respectively. The dashed and dotted line represents a feasible singular path, and lines of small dots represent infeasible solutions. The solutions to the optimization problem need not be points of minimum potential energy $E(d, h; p)$, which is not minimized on the segments from (b) to (c) and from (d) to (c) to (e). However, all other feasible path segments do correspond to physical states of the truss where the potential energy is minimized.

We now describe these singularities and the connecting path segments, beginning with those which occur along the solution branch where the constraint $h \leq 1.5$ is active. Loss of strict complementarity gives rise to the bifurcation points (g), (a), and (c), whose presence was indicated by a change in sign of the multiplier λ_a. At these points the inequality constraint becomes weakly active and solution paths bifurcate into the region $0 < h < 1.5$. The fold points (b) and (d) ($p = \pm.37$), which resulted from violation of the linear independence constraint qualification, were detected by a change in the sign of ν. The type of the solution along this solution branch is determined by the sign of λ_a/ν, which changes at each of these five singular points. This results in the alternating segments of minimizers and maximizers shown in Figure 2.

The solution to the parametric design problem can now be described for $p > 0$. Given a small but positive load p, the global minimum occurs on the branch of minimizers between singular points (f) and (a). As the load p is increased from zero, the height h increases from $\sqrt{2}$ to $h = 1.5$ where the constraints $h \leq 1.5$

becomes active. As the load p is increased further, the deflection d continues to increase along the path from (a) to (b) until p reaches 0.37 where the truss 'snaps through' and there is no minimum beyond $p = 0.37$ corresponding to a height h near 1.5. (The only way to maintain an optimum locally beyond $p = 0.37$ is to increase the parameter $\beta = 1.5$ in the upper bound on the height h.) The local minimizer corresponding to $h = 0$ becomes the global minimizer for p beyond $p = 0.37$. Local sensitivity is surely present at points (a) and (b). Note that the path of minimizers is continuous but not differentiable at (a). (Near such points many optimization codes exhibit cycling.) At the fold point (b), the path of minimizers ceases to exist. Optimization codes would have difficulty here since the unnormalized multipliers will be large and go to infinity as p approaches 0.37. The conclusion with regard to the design of the truss is that for stability the loads must be less than $p = 0.37$ and that sensitivity occurs near the singular points (a) and (b) for the reasons stated. Clearly, the ability of the continuation procedure to locate such singular points and obtain such a global analysis is a major strength of the methodology.

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VII.. Technical Information for the 91-92 Contract

VII.A. Editorships Associator Editor of *Computational Optimization and Applications*

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VII.C. Papers

1. Variable Order Adams-Bashforth Predictors with Error-stepsize Control for Continuation Methods, *SIAM Journal on Scientific and Statistical Computing,* Vol. 12, No. 3, May, 1991 pp 695-723 (with Bruce N. Lundberg).
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10. Numerical Continuation and BIfurcation Techniques in Parametric Nonlinear Programming, to appear in *Proceedings of the Fourth AIAA/USAF/NASAOAI Symposium on Multidisciplinary Analysis and Optimization, 1992* (with Bruce N. Lundberg).
11. Optimality, Stability and Convergence in Nonlinear Control, submitted to *Applied Mathematics and Optimization*, with A. Dontchev, W. W. Hager, and B. Yang.
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14. Data association problems posed as multidimensional assignment problems: problem formulation, in O. E. Drummond, ed., *Signal and Data Processing of Small Targets*, SPIE, Vol. 1954, Bellingham, WA, 1993, N. Rijavec, M. Liggins, and V. Vannicola.
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17. Bifurcations in Abstract Constrained Optimimization, in preparation.
18. A Lagrangian Relaxation Algorithm for a Class of Multidimensional Assignment Problems, in preparation, to be submitted to COAP, with Nenad Rijavec
19. A Comparison of Several Algorithms for Multidimensional Assignments, submitted, 1993, with N. Rijavec.

VII.D. Lectures

1. Multitarget Tracking on Parallel Processors, IBM-FSD, Owego, New York, January, 1991.
2. Multitarget Tracking and Multidimensional Assignment Problems, SPIE Meeting, Orlando, FL, April, 1991.
3. Parametric Problems in Constrained Optimization and Control, University of Florida, Gainesville, FL, April, 1991.
4. The Data Association Problem in Multitarget Tracking and Multidimensional Assignment Problems, SDIO Parallel Programmers Meeting, Falcon AFB, Colorado Springs, April 30, 1991.
5. Multi-Target Tracking on Parallel Processors, Space Tech, Fort Collins, Colorado, May 20, 1991.
6. Multitarget/Multisensor Tracking and Multidimensional Assignment Problems, US Army SDC, Hunstville, Ala., June 10, 1991.
7. Multitarget Tracking and Multidimensional Assignment Problems, SDIO Meeting Hunstville, Ala., June 13, 1991

8. Multitarget/Multisensor Tracking and Data Association Problems, AFOSR/ Rome Lab Workshop on Multisensor Fusion, Rome, NY, June 18, 1991.
9. Parametric Problems in Constrained Optimization and Control, Conference on "Numerical Optimization Methods in Differential Equations and Control," NCSU, Raleigh, NC, July, 1991.
10. The Data Association Problem in Multitarget/Multisensor Tracking, IBM, Boulder, CO and Martin Marietta, Denver, CO, August, 1991.
11. Numerical Continuation Methods in Parametric Nonlinear Programming, International Conference on Parametric Optimization and Related Topics III, Güstrow, Germany, August, 1991.
12. Some New Approaches to Multitarget/Multisensor Tracking, Army Research Office, Research Triangle Park, November 15, 1991.
13. Numerical Continuation and Bifurcation Methods in Constrained Optimization, University of Florida, Jan., 1992.
14. The Data Association Problem in Multitarget/Multisensor Tracking, University of Florida, Jan., 1992
15. Multitarget/Multisensor Tracking, Rome Labs, Griffiss AFB, Feb., 1992
16. Multitarget/Multisensor Tracking, Federal Sector Division of IBM, Feb., 1992
17. Multitarget/Multisensor Tracking, DARPA, Washington, DC, March 25, 1992
18. Data association for track initiation and extension using multiscan windows, 1992 SPIE Conference on Small Targets, Orlando, FL, April, 1992.
19. Data association for track initiation and extension using multiscan windows, Los Alamos Days at Boulder, CU-Boulder, April, 1992.
20. Data association for multisensor data fusion and multitarget tracking, Grumman Corporation, Long Island, NY, July, 1992.
21. Data association for multisensor data fusion and multitarget tracking, Rome Labs, Griffiss AFB, July, 1992.
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23. Numerical Continuation and Bifurcation Techniques for Parametric Nonlinear Programming, Fourth AIAA/USAF/NASA/OAI Symposium on Multidisciplinary Analysis and Optimization, Cleveland, Ohio September 22, 1992.